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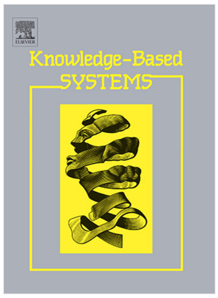

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AN IMPROVEMENT TO SWING TECHNIQUES FOR ELICITATION IN MCDM METHODS

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Abstract: Several approaches that utilise various questioning procedures to elicit criteria weights exist, ranging from direct rating and point allocation to more elaborate methods. However, decision makers often find it difficult to understand how these methods work and how they should be comprehended. This article discusses the SWING family of elicitation techniques and suggests a refined method: the P-SWING method. Based on this, we provide an integrated framework for elicitation, modelling and evaluation of multi-criteria decision problems.

Keywords: Multi-criteria decision methods; Weight elicitation; Improved SWING method, Quality assurance.

1. Introduction

Although promising from a decision-theoretical perspective, formal and semi-formal decision methods such as multi-criteria decision methods (MCDM) remain rather uncommon in real-life decision modelling and analyses. This seems to owe at least to some extent to perceived difficulties in understanding the decision models available. In particular, there exist several methods and approaches designed to elicit criteria weights that utilise various questioning procedures, ranging from direct rating and point allocation to more elaborate methods. Numerous methods use trade-offs in a structured manner, with significant effects for actual decision-making. However, decision makers continue to find it difficult to understand their own preferences and how these correspond to the elicitation methods used for this purpose. Furthermore, most decision information is imprecise, rendering many prevalent decision tools inappropriate in the sense that they cannot inherently represent uncertainties. Some decision methods allow for the modelling of imprecision, in particular ordinal rankings and interval approaches (both for criteria weights and values), with the aim of avoiding unrealistic, overprecise or even meaningless statements, and instead only demanding information that the decision maker is able to express with confidence. Many MCDM researchers have thus argued that unreasonable exactness is counterproductive and that other means are necessary. Preference rankings appear to constitute one of the most commonly used means in this regard.[2]

There are consequently a multitude of approaches to express preference intensities, such as the MACBETH method (Bana e Costa et al., 2002), ranking using the delta-ROC (Rank Order Centroid) approach (Sarabando and Dias, 2010), or more simplified methods such as Simos' method and varieties (Figueira and Roy, 2002). The Smart Swaps methods also exist (Mustajoki and Hämäläinen, 2005), while Jiménez et al. (2006) combine various techniques in the GMAA system. Elicitations are based on attribute trade-offs or by directly assign weight intervals. These relaxations of precise judgments are understood to model decision problems more realistically (see e.g. Larsson et al., 2014; Park, 2004). However, solutions to such problems are sometimes hard to find and the results can be difficult to interpret. Numerous suggestions have also been made over the years, based on (for example) sets of probability measures, upper and lower probabilities, interval probabilities and utilities (Utkin, 2017), fuzzy measures (Aven and Zio, 2011; Shapiro and Koissi, 2015; Tang et al., 2018) and

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[2] See e.g. Barron and Barrett (1996), Riabacke et al. (2012) and Danielson et al. (2014) for extensive discussions of elicitation procedures, including issues regarding precision.

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evidence and possibility theory (cf., e.g. Dubois, 2010; Dutta, 2018; Rohmer and Baudrit, 2010). There are also approaches based on second-order techniques (Danielson et al., 2007; Ekenberg et al., 2014). Other approaches modify some classical decision rules, such as the central value rule based on the midpoint of the range of possible performances (cf. Aguayo et al., 2014; Ahn and Park, 2008; Mateos et al., 2013; Sarabando and Dias, 2009). Salo and Hämäläinen (2001) have suggested a set of approaches for handling imprecise information in these contexts, such as the PRIME method for preference ratios, while the SMART method has also been implemented in software (see e.g. Mustajoki et al., 2005). Nevertheless, these approaches exhibit various difficulties, including combining both interval and qualitative estimates with weighted decision rules but without introducing very rough evaluation measures such as Γ-maximin or (Levi's) E-admissibility (cf., e.g. Augustin et al., 2014). Greco et al. (2008) suggest UTA^{GMS} for a purpose similar to this paper (which uses an ordinal regression technique), generating a representation extracted from pairwise comparisons even when ordering is incomplete. Figueira et al. (2009) generalise this by taking cardinalities into account in order to obtain a class of total preference functions compatible with user assessments, restricting the polytope in various respects. For our purposes, this is less suitable because it is unclear how it can be extended when other types of information (such as interval constraints) also exist, resulting in computational issues as explained in, for instance, Danielson and Ekenberg (2007). Furthermore, in many cases the structural constraints can be represented by second-order information (Ekenberg et al., 2005), which provides further information that should be handled. Hence, our representation is in such respects more appropriate to the purpose of this paper, as explained below. In any case, the formalism suggested is by no means the only possibility, and should instead be considered an example (as well as being the foundation for the computer tool used below).

One of the most important problems in many MCDM methods is the handling of trade-off effects between the value scales of different criteria. Trade-off methods are quite useful, but given the number of judgements required of the decision maker they can also be very demanding and sometimes intractable. For example, Fischer (1995) highlights that trade-off methods tend to give greater weight to the most important attribute. One prominent family of methods addressing this and other problems is SWING weighting (von Winterfeldt and Edwards, 1986). As an example, the popular SMART family of MCDM methods was extended with SWING trade-offs, yielding the SMARTS method (Edwards and Barron, 1994).

This article suggests a refined method – the P-SWING method – in an attempt to overcome some of the typical problems associated with elicitation. The method consists of an amended swing-type technique at its core. However, whereas a traditional SWING session only contains from-worst-to-best swings, the suggested method adheres to the core ideas while allowing for intermediate comparisons as well. This will aid the convergence of the weights for the criteria. Furthermore, there is no use of zero alternatives or similar synthetic constructs, and instead many more available real data points are utilised. Based on this, we provide an integrated framework for elicitation, modelling and evaluation of multi-criteria decision problems.

The following section describes an experiment to compare different MCDM methods in which some problems with SWING techniques were detected as side effects, and subsequently explored alongside remedies via focus groups. In section 3, we formalise these remedies into an extended method for criteria weight elicitation with improved precision, called P-SWING (Partial SWING). Section 4 describes how P-SWING is integrated into a framework for elicitation, modelling and evaluation of multi-criteria decision problems. Sections 5 and 6 then describe in detail how the framework is used in practice, in order to demonstrate its advantages. Finally, section 7 concludes the paper.

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2. MCDM methods

In order to investigate how some popular classes of MCDM methods are perceived and used in real-life decision making situations, we conducted a study involving 100 people making one large real-life decision each (Danielson and Ekenberg, 2016). A requirement was that such a decision was important, not obvious to the decision maker, and required substantial information collection in advance. The decisions included selecting a country or area in which to live, choosing a university programme and buying an apartment. The three classes of methods studied were generalisations of some of the most popular MCDM methods, i.e. three very common classes of value function methods:

- proportional scoring methods, such as the SMART family of methods;
- pairwise ratio scoring methods, such as the AHP method; and
- cardinal ranking methods, such as the MACBETH or CAR methods.

Both the proportional scoring and the cardinal ranking methods were supported by a SWING procedure in the step whereby criteria weights were elicited from the decision makers.

2.1 Initial study

As discussed in Danielson and Ekenberg (2016), each individual in a group was offered two to three weeks to complete a decision-making task using the three methods in parallel, before being asked to reflect on the advantages and disadvantages of each method. In order for the results to be comparable, the methods were supported by computer tools with very similar user interfaces, ensuring that the three methods were applied correctly. Adequate tutoring and guidelines for each method were available throughout the decision-making processes. The decision makers' respective reports contained decision data as well as results from and a comparison of all three methods. The participants were subsequently interviewed in focus groups and their results regarding the respective methods were analysed and compared.

However, while the results demonstrated that cardinal ranking methods outperformed scoring methods and pairwise comparing methods (both in terms of actual simulation results and the participants' issues with using the respective methods), a complication was later discovered in the concluding focus groups in which each participant discussed his or her work. Indeed, during the focus group discussions it became evident that a large number of the participants had not fully understood the concept of swing weights in spite of having received ample instructions before as well as guidance during the work. The misunderstanding did not affect any method in particular (rather, the confusion was more of a general nature), but it was apparent that many participants treated swing weights as if they were absolute (a priori) weights not tied to the particular attribute scales in question. This may invalidate the outcomes of the usage of any decision method employing relative weights, and thus represents a serious obstacle to the widespread use and acceptance of decision analytical methods in general.

2.2 Enlarged study

Given that the study in Danielson and Ekenberg (2016) was not designed to deal with this issue, we subsequently conducted a study with 39 new participants, asking them to estimate absolute (a priori) weights for their criteria before the work began. They were also told to use relative swing weights during their decision work. After their decisions were made, the decision processes for the determination of criteria weights were discussed in focus groups. The subjects were then assessed according to whether they were able to differentiate between absolute (a priori) and relative (swing) weights. Three indicators were used: how close the relative weights were to the absolute, whether the relative weights were modified when alternatives with a large impact on some scales were introduced, and the reasoning when the relative weights were determined. Of the 39 participants, only four demonstrated a clear understanding of the difference between absolute and relative weights. If this result is indicative of a wider (mis)use of relative weights, a SWING based methodology seems to be insufficient when eliciting criteria weights. On the other hand, absolute weights are neither mathematically nor logically advisable and also cause severe difficulties when calibrating scales. However, from the focus group discussions, one important observation was possible: a commonality between those who had realised the difference between relative and absolute weights and those who could realise it after the discussions was that they were able to comfortably reason about subparts of the scales where real decision objects (alternatives) were positioned. This implies a third elicitation option: to use a modified relative weight elicitation technique.

During the study, it was observed that contrived reference objects such as made-up best or worst cases or "zero alternatives" constituted particularly poor vehicles for thought. Many participants exhibited considerable difficulty in understanding them or their meaning. Subsequent discussions in the focus groups converged into two observations on desirable properties (in addition to a swing-like procedure) for an elicitation technique to possess:

1. The focus during the elicitation should only be on the existing real-life alternatives without any abstract additions.
2. When constructing the ordering of the criteria weights, the procedure should not be limited to extreme points (the endpoints of the value scales), but should rather allow the use of all values actually asserted.

Based on these desiderata and on discussions in the focus groups regarding the ways in which remedies and solutions could be introduced, we have designed an elicitation technique that extends the SWING methodology by introducing partial assignments and interval constraints. This extension is applicable to all SWING-related methods and has been coined P-SWING (Partial SWING), which is formalised in the following section and then exemplified by extending an existing MCDM method.

To recap, cardinal ranking methods (represented by the CAR method) were superior to other classes of methods, but the elicitation component could be improved. We therefore propose the P-SWING method, consisting of an amended swing-type technique at its core. The basis is that while a traditional SWING session embraces only from-worst-to-best swings, P-SWING employs intermediate comparisons as well. This will rapidly aid the convergence of the weights for the criteria. Furthermore, there is no use for zero alternatives or similar synthetic constructs, and instead many more real data points are utilised. In order to enable a stability analysis during the evaluation phase, we also introduce intervals around the surrogate weights generated from the elicitation process.

3. P-SWING

Modelling realistic decision problems often results in numerically imprecise and vague sentences, such as *"the value of alternative A_1 under criterion C_1 is greater than 40 %"* or comparative sentences such as *"the value of alternative A_1 under criterion C_1 is preferred to the value of alternative A_2 under criterion C_1."* Such sentences are easily translated into a numerical format. In the interval case, the translation is of the format $v_{ij} \in [a_1, b_1]$, i.e. the two linear inequalities $v_{ij} \geq a_1$ and $b_1 \geq v_{ij}$, where a_1 and b_1 are real numbers on the scale under consideration. Similar translations apply when representing comparative sentences, where we attain inequalities in the format $v_{ij} \geq v_{kl}$. More generally, the statements of the decision makers are represented by linear inequalities involving a set of decision variables $\{x_i\}$, $i \in I$, which can

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be translated into the format $k_1x_1 + k_2x_2 +\ldots+ k_nx_n$ ¤ b for some constants k_i, $\forall i\in I$, and b, as well as relational operators ¤ representing equalities or strict or weak inequalities.[3]

3.1 The P-SWING process

Assume that values for each attribute A_i under each criterion C_j have been elicited. The ensuing step will be to assign weights to the criteria such that $\sum_j w_i = 1$. The P-SWING procedure is then carried out in two steps as follows. The basic idea is that after the ordinary weight comparisons have been undertaken, a further step is added for the purpose of verifying that the initial ranking is preserved, i.e., an indication that the decision maker is aware of what he or she is expressing. However, another important feature here is to provide the possibility to increase the precision in the estimate by comparing subscales with one another. The P-SWING procedure steps are:

a) In a rather traditional swing-type session, the decision maker is asked to compare the swings between the endpoints (best and worst outcome) regarding the criteria's respective value scales. The criteria weights are ranked using an ordinal ranking function amended with '='. Questions asked are of the type *"Which is the most important to you: the difference between endpoints in criterion C_i or in criterion C_j?"* The result of this step might (for instance) be a ranking $w_1 > w_2 = w_3 > w_4 = w_5$, or numerical scores if such a weight representation is being used.

   Note that if we assume that v_{i0} and v_{i1} are the endpoints of the value scale for criterion C_i, the comparisons are then of the type $(v_{i1}-v_{i0})\cdot w_i > (v_{j1}-v_{j0})\cdot w_j$, i.e. of the character of the ordinary comparisons $w_j > w_i$.

b) The baseline of the next step is that fractions of the criteria's respective value scales are compared. Questions asked are now of the type *"Which is the most important to you: the difference between the values α_2 and α_1 in criterion C_i or between the values α_4 and α_3 in criterion C_j?"* This step thus introduces a new feature by allowing to compare parts of the scales with one another.

   The statements then consequently become of the type $(\alpha_1\cdot v_{i1}-\alpha_2\cdot v_{i0})\cdot w_i > (\alpha_3\cdot v_{j1}-\alpha_4\cdot v_{j0})\cdot w_2$ for real value statements α_1 to α_4 in [0,1], where $\alpha_m\cdot v_{i1}-\alpha_n\cdot v_{i0} > 0$, for all *i, n, m*. We call these statements *α-statements*.

   This also means that the questions only focus on real alternatives existing in the current decision context. In this way, a revised system of inequalities (and equalities) is formed, and if this system has a solution, it is consistent, i.e. the decision maker has made a consistent assessment of the relative importance of different criteria. The weights are adjusted in accordance with the new system.

Each statement is thus represented by one or more constraints, and after a session we receive two sets of linear constraints: one containing the values of the alternatives under the respective criterion and one containing the weight statements.

3.2 P-SWING evaluations

In order to facilitate the execution of a P-SWING process, there must be procedures present to continuously validate the input and support further input. In this section, we suggest a formalism that will take care of this support by introducing and ensuring consistency in two sets of linear constraints: one set of weights (the ones to be swinged) and one set of values

[3] The index set I is {1,…,*n*} where *n* is the number of variables in X.

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(the ones to form the judgement basis for swinging). This will help prepare the evaluation of the decision problem, as it consists of evaluating the formula (1) (see section 4 below) involving the weights and the values of the problem.

In the presentation below, we will refer to the conjunction of constraints for the weights, together with $\Sigma_i w_i = 1$, as the *swing base* (S). The *value base* (V) consists of similar translations of vague and numerically imprecise value estimates in terms of v_{ij}. The collection of alternatives, criteria as well as the weight and value statements constitutes a *decision problem*. Furthermore, the initial most representative point (MR-point) of the weights must be modified according to the new information provided.

Definition 3.1: Given a set of variables $S = \{x_i\}$, $i \in I$, a continuous function $g: S^n \rightarrow [0, 1]$, and real numbers $a, b \in [0,1]$ with $a \leq b$, an *interval constraint* $g(x_1,\ldots,x_n) \in [a,b]$ is a shorter form for a pair of weak inequalities $g(x_1,\ldots,x_n) \geq a$ and $g(x_1,\ldots,x_n) \leq b$.

In this manner, equalities and inequalities can be handled in a uniform way. There are many types of constraints, and they correspond to different types of decision-maker statements.

Definition 3.2: Given a set of variables $\{x_i\}$, $i \in I$, and real numbers $a, b \in [0, 1]$ with $a \leq b$: A *comparative constraint* is an interval constraint of the form $x_i - x_j \in [a,b]$ with $i,j \in I$ and $i \neq j$.

All interval constraints are linear. A collection of interval constraints concerning the same set of variables is called a constraint set, and it forms the basis for the representation of decision situations.

Definition 3.3: Given a set of variables $\{x_i\}$, $i \in I$, a *constraint set* in $\{x_i\}$ is a set of interval constraints in $\{x_i\}$.

From the definition of an interval constraint, it follows that a constraint set can be seen as a system of inequalities. For a system of inequalities to be meaningful, there must be some vector of variable assignments that satisfy each inequality in the system simultaneously.

Definition 3.4: Given a set of variables $\{x_i\}$, $i \in I$, a *solution* to a system X of inequalities in $\{x_i\}$ is a real vector $\mathbf{a} = (a_1,\ldots,a_n)$ where each a_i is substituted for x_i such that every inequality in the system is satisfied[4]. The vector **a** is called a *solution vector* to X. The *solution set* for X is $\{\mathbf{b} \mid \mathbf{b}$ is a solution to X$\}$.

Constraint sets have many properties in common, whether they are weight or value constraint sets. The first question is whether the elements in a constraint set are at all compatible with one another. This translates to the problem of whether a constraint set has a solution, i.e. if there exists any vector of real numbers that can be assigned to the variables.

Definition 3.5: Given a set of variables $\{x_i\}$, $i \in I$, a constraint set X in $\{x_i\}$ is *consistent* **if** the system of weak inequalities in X has a solution.[5] Otherwise, the constraint set is *inconsistent*. A constraint Z is *consistent with* a constraint set X **if** the constraint set $\{Z\} \cup X$ is consistent.

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[4]There exists a solution if the substitution of a_i for x_i in X, for all $1 \leq i \leq n$, does not yield a contradiction.

[5]Hence there is a non-empty solution set for X.

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In other words, a consistent constraint set is a set where the constraints are at least not contradictory.

Definition 3.6: A *swing base* S consists of a set of swing weight statements to which $\sum_j w_j = 1$ is added.

Definition 3.7: A *swing decision problem* contains the following information about a decision situation:

- A set of alternative courses of action $\{A_i\}$ for i = 1,…,m ($m \geq 2$);
- A set of criteria $\{C_i\}$ for i = 1,…,n ($n \geq 2$);
- For each alternative A_j and each criterion C_i, a value v_{ij} on a value scale for that criterion;
- A swing base S containing all swing statements.

According to the definition of an interval statement, a base can be seen as a set or system of inequalities. The first question is whether the statements in a swing base are compatible with one another. This translates into a question of pointwise consistency.

Definition 3.8: A *solution* to a swing base is a vector $\mathbf{w} = (w_1,\ldots,w_m)$ such that every equation in the corresponding system is satisfied.

Definition 3.9: A swing base is *pointwise consistent* (or p-consistent for short) **if** there exists at least one solution to the base. Otherwise, the base is p-inconsistent.

In other words, a p-consistent swing base is a base where the translated statements are at least not contradictory. This is a required property for a swing base following completion of the P-SWING procedure.

However, pointwise consistency constitutes a rather weak property of a swing base. If the statements in the base are consistent only at a single point, the base is vulnerable to small changes in the input data and to the effects of sensitivity analyses. Given that we are working with high degrees of imprecision, this property alone is thus too weak. We must be assured that the base would remain consistent at least for reasonably small changes in the interval statements.

Single-point solutions in the bases are thus essentially meaningless and, to make the concept of consistency stronger, we introduce the concept of regular consistency.

Definition 3.10: A consistent base X with variables $x_1,\ldots,x_n$ is regularly consistent (or r-consistent for short) relative to a given regularity vector $r = (r_1,\ldots,r_n)$ if for each component in the norm $(d_1,\ldots,d_n)$, $d_i \geq r_i$. The r_is are called regularity values.

It is convenient to discuss properties of a single equation or interval statement added to a base.

Definition 3.11: An equation or interval statement Z is *r-consistent with* an r-consistent base X **if** the base $\{Z\} \cup X$ is r-consistent.

Definition 3.12: A decision problem is *r-consistent* **if** the value base and the swing base are both r-consistent.

The most fundamental computational component in P-SWING is a way of calculating the consistency of a swing base. Given that the base consists of a linear system of interval equations, the natural candidate for an algorithm is linear programming. In fact, p-consistency is equivalent to completing phase I of a standard linear programming (LP) problem. As noted above, a swing base is pointwise consistent if any solution can be found to the set of interval equations. Let there be *m* interval equations in the base. By introducing new variables $y_1,\ldots,y_k$, with $k = 2 \cdot m$, to the consistency problem, it can be reformulated as

$$\begin{aligned}&\min\,(y_1 + \ldots + y_k)\\&\text{when } \mathbf{Ax} \geq \mathbf{b}\\&\text{and } \mathbf{x} \geq \mathbf{0},\ \mathbf{y} \geq \mathbf{0},\end{aligned}$$

where each interval equation $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \in [b_i,d_i]$ is transformed into the two equations $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - y_j \geq b_i$ and $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + y_l \leq d_i$. If the obtained minimum of $y_1 + \ldots + y_k$ has the value **zero**, then a solution has been found that does not contain any y_i. Removing the y_is, the resulting solution vector **x** is indeed a feasible solution, that is, the base is proven to be consistent. If the minimum of $y_1 + \ldots + y_k$ is positive, then it is certain that the optimal values of the y_is are larger than zero, that is, at least one of the y_is is necessary to keep the base consistent. Given that the y_is were added to the base, the problem itself has no solution. Hence, the base is inconsistent.

4. Evaluation

The evaluation process is uncomplicated to perform. Assume a standard MCDM method that seeks to evaluate each alternative, yielding a most representative point (MR-point[6]) for each alternative. First, we make a pre-elicitation as in step a) of P-SWING and calculate the MR-point with a suitable interval environment around it. Thereafter, the α-statements are added. These calculations are made by the LP-algorithm above. If we still have an r-consistent decision problem, we can proceed. The adjusted MR-point is the point that has the least distance from the original MR-point, as expressed by the definition below.

Definition 4.1: Given an r-consistent decision problem in *n* dimensions, assume that the extreme points in each projection on the axes of the orthogonal base of the system are $[a_i, b_i]$, and that the MR-point for that dimension is $\bar{c}= (c_1,\ldots,c_n)$, then the *adjusted MR-point*, $\bar{c}'= (c_1',\ldots, c_n')$, is

$$\mathrm{argmin}_{\bar{c_i}'} \sum_i^n \left(\frac{c_i - a_i}{b_i - a_i} - \frac{c_i' - a_i'}{b_i' - a_i'}\right)^2.$$

Following the elicitation phase, the multi-criteria decision problem is evaluated as a multi-linear problem against the background information contained in the r-consistent decision problem and the adjusted MR-point. This means that we solve equations of the format

$$E(A_i) = \sum_{i_1=1}^{n_{i_0}} x_{ii_1} \sum_{i_2=1}^{n_{i_1}} x_{ii_1i_2} \cdots \sum_{i_{m-1}=1}^{n_{i_{m-2}}} x_{ii_1i_2\cdots i_{m-2}i_{m-1}} \sum_{i_m=1}^{n_{i_{m-1}}} x_{ii_1i_2\cdots i_{m-2}i_{m-1}i_m} x_{ii_1i_2\cdots i_{m-2}i_{m-1}i_m}, \quad \mathbf{(1)}$$

given r-consistent decision problems. The expected values $E(A_i)$ are computed by solving successive linear programming problems in each base (weight and value). Given that the weight and value bases are independent, the collected solutions constitute the total solution to the multi-linear problem in (1).

[6] An MR-point is the most representative point that represents a solution to the problem. If probabilities are involved, this is usually the expected value. If criteria weights are involved, this is the weighted value over all criteria and thus over all value scales. The MR-point is a general concept covering all of the above situations and combinations thereof.

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Over the years, we have developed processes and software libraries to solve problems of this type in a more general way, by expanding a Multi-Attribute Utility Theory (MAUT) approach that allows for imprecise estimates of various types. One example is the software DecideIT, which allows for imprecision of the kinds that we have in r-consistent decision problems with numerically imprecise weights and values. The cardinal ranking of DecideIT compares the performance of each alternative to others as well as providing an estimate of the reliability of the result. This tool considers the entire range of values as alternatives present across all criteria, and displays the plausibility of an alternative outranking those that remain. Various versions of DecideIT have been used in a wide variety of contexts, such as infrastructure development, long-term storage of nuclear waste, choice of insurance portfolios, demining, gold mining and applications for financial risks (Danielson and Ekenberg, 2007; Danielson et al., 2007, 2009; Ekenberg et al., 2009, 2017; Mihai et al., 2015).

The basic function of DecideIT is to investigate the ranges of values and weights for which a strategy is optimal against a set of equations, for instance of the type $v_{11} > v_{21}$, $w_1 > 0.1$, $w_1 > 0.3$, $w_1 > w_3$, $w_1 \in [0.3, 0.7]$, $v_{11} \in [0.5, 0.6]$, etc. By examining the number of assignments of variable values to which the different strategies are superior or inferior, respectively, we can investigate the properties of the strategies. A detailed account of DecideIT and the utilisation of second-order information are beyond the scope of this article,[7] but below we present an example to illustrate how to use DecideIT together with P-SWING.

5. Example of P-SWING evaluation process and use

Consider a procurement process in which a large organisation is looking for a new office space, as its existing space has become less adequate. The decision situation is to select a space from four real estate developers, *A*, *B*, *C*, and *D*, in order to realise this project. The criteria emphasised in the selection process are *functionality* (basically the degree of adequacy of the new premises), *localisation* (geographical and infrastructural), *opportunities for interaction with the surrounding society*, and *price*.

Elicitation

First, the values for the alternative providers (when taking all participants' preferences into account) are summarised, as below. We set the qualitative scales as [0, 1] and let the scale for the price be the actual price.

Functionality	***Localisation***	***Opportunities***	***Price***
A is better than B	B is slightly better than C	B is better than A	A costs 5.5 MEUR
B is slightly better than C	C is better than A	A is better than C	B costs 6.0 MEUR
C is better than D	A is better than D	C is better than D	C costs 5.0 MEUR
			D costs 4.0 MEUR

We express this in a semantics using '$>_i$' symbols for denotation:[8]

$>_0$ equally good
$>_1$ slightly better
$>_2$ better
$>_3$ much better,

[7] See Ekenberg et al. (2017) and Danielson and Ekenberg (2018) for details.

[8] Needless to say, there are various suggestions for how to interpret such statements (cf., e.g. Xu, 2013; Chen and Hong, 2014), but we will not discuss the exact wordings and their possible semantics, as interpretations are considered geometrically. If other candidates were considered more reasonable for one reason or another, the number of steps between the discriminative statements could be changed without affecting the general idea.

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where $x_k >_i x_{k+1}$ is $x_k > x_{k+1}$ when i = 1 and $\{x_k > x_{k_1}, x_{k_1} > x_{k_2} \ldots, x_{k_{i-1}} > x_{k+1}\}$, i.e., a set of linear expressions connoting i "steps" between x_k and x_{k+1}, using auxiliary variables x_{k_i}, when i > 1.

This results in the following value statements:

$v_F(A) >_2 v_F(B)$	$v_L(B) >_1 v_L(C)$	$v_O(B) >_2 v_O(A)$	$v_P(A) = 5.5$
$v_F(B) >_1 v_F(C)$	$v_L(C) >_2 v_L(A)$	$v_O(A) >_2 v_O(C)$	$v_P(B) = 6.0$
$v_F(C) >_2 v_F(D)$	$v_L(A) >_2 v_L(D)$	$v_O(C) >_2 v_O(D)$	$v_P(C) = 5.0$
			$v_P(D) = 4.0$

Following the process described above, and assuming that there are no immediate conflicts in the initial preferences, they make up an initial ranking that results in *Functionality* being the most important criterion, followed by *Localisation*. Thereafter follows *Opportunities*, and finally *Price*.

Considering the scale endpoints, assume that the participants provide the following statements as a result of step (i), yielding the following initial ranking: *Functionality* is slightly more important than *Localisation*, which is more important than *Opportunities*. Finally, *Opportunities* is more important than *Price*. This is translated into the following cardinal ranking order:

- $w(F) >_1 w(L)$
- $w(L) >_2 w(O)$
- $w(O) >_2 w(P)$

In step (ii), the decision makers react by providing the following supplementary statements for the criteria:

- The difference between B and C in *Functionality* is more important than B and A in *Localisation*.
- The difference between C and D in *Functionality* is more important than A and D in *Opportunity*.
- The difference between C and A in *Localisation* is more important than B and D in *Opportunity*.
- The difference between B and C in *Localisation* is more important than a *Price* difference of 1 MEUR.

Evaluation

In spite of the structural simplicity of the problem, it is comparatively difficult to provide a recommendation without further analysis. The value statements are measured on [0, 1]-scales by assigning '1' to the best value and '0' to the worst in each criterion. The other values are henceforth placed linearly on each [0, 1]-scale so that each "step" in the description above occupies an equally wide interval and the sum of the intervals fully cover the [0, 1]-scale.[9] The only exceptions are the endpoints where the intervals do not extend beyond the points '0' or '1'.

Criterion *Functionality*:

- [9] For example, assume that the statements are $w(X) >_1 w(Y)$ and $w(Y) >_3 w(Z)$. This yields 4 steps in total, with each step ¼ in size on the [0, 1] scale. X is placed at the upper end (1) and Z is placed at the lower end (0). Y is now placed 1 step from the top and 3 steps from the bottom, at 0.75.

	Lower bound	Upper bound
A	0.900	1.000
B	0.500	0.700
C	0.300	0.500
D	0.000	0.100

Criterion *Localisation*:

	Lower bound	Upper bound
A	0.300	0.500
B	0.900	1.000
C	0.700	0.900
D	0.000	0.100

Criterion *Opportunities*:

	Lower bound	Upper bound
A	0.583	0.750
B	0.917	1.000
C	0.250	0.417
D	0.000	0.083

Criterion *Price*:

	Lower bound	Upper bound
A	0.125	0.375
B	0.000	0.125
C	0.375	0.625
D	0.875	1.000

Thereafter, we calculate the r-consistent decision problem from the initial rankings. The feasible region (orthogonal hull) of the criteria weights is then computed in two steps. First, the MR-point is calculated using the CAR method (Danielson and Ekenberg, 2016). To cater for the inherent imprecision in the elicited information, an interval of about ±10% is subsequently formed around the MR-points by way of the DecideIT software implementation of CAR. This yields the following weights:

	Lower bound	MR-point	Upper bound
w(F)	0.396	0.453	0.553
w(L)	0.264	0.302	0.369
w(O)	0.145	0.170	0.198
w(P)	0.038	0.075	0.098

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The supplementary statements in step (ii) in the P-SWING process are translated as (refer to the information above):[10]

- 0.4w(F) > 0.6w(L)
- 0.4w(L) > w(O)
- 0.4w(F) > 2/3 w(O)
- 0.2w(L) > 0.5w(P)

After the statements in step (ii) have been considered, the orthogonal hull has shrunk but remains valid (i.e. non-empty), which continues to provide a consistent system and furthermore indicates that the decision maker(s) have understood the relative nature of the criteria weights.

The modified weight intervals and adjusted MR-point are then the following:

	Lower bound	Adjusted MR-point	Upper bound
w(F)	0.436	0.480	0.553
w(L)	0.264	0.283	0.327
w(O)	0.145	0.163	0.186
w(P)	0.038	0.074	0.098

A criteria tree containing this information is shown in Figure 2.

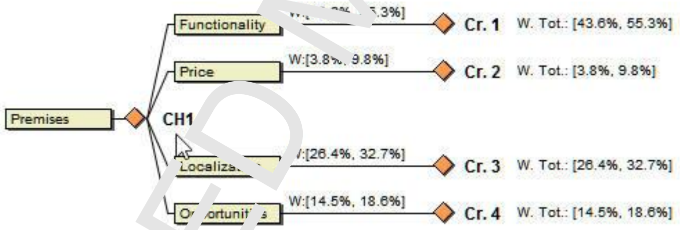

Figure 2. Criteria tree.

Once modelled, the problem can be evaluated. We use formula (1) above by using the DecideIT[11] tool for analysis. In so doing, we can attain greater information regarding the factors involved. An initial result can be seen in Figure 3.

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[10] For example, the statement "*The difference between B and C in Functionality is more important than B and A in Localisation*" by the decision maker entails that the difference between B and C on the Functionality scale (0.4) carries greater importance to the decision maker than the difference between B and A on the Localisation scale (0.6). This is then entered into the system of equations and inequalities as 0.4·w(F) being greater than 0.6·w(L). These added inequalities form a set of anchor frames that the expected value solutions may not violate.

[11] The P-SWING algorithms in this paper are implemented in the DMC decision library that underlies the DecideIT tool.

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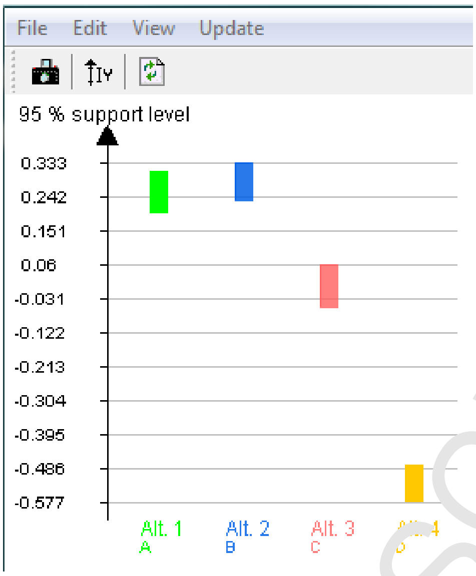

Figure 3. A first evaluation of the decision situation.

In the figure, the software displays the result of assigning all possible values to all variables, given the supplied intervals and relations. Thus displayed are all possible expected value ranges (minimum through maximum) given the information entered. The figure illustrates how the strategies (alternative courses of action) relate to one another given the values defined through our ranges and comparisons. The green bar represents provider *A*, the blue bar represents provider *B*, the red bar represents provider *C*, and finally the yellow bar represents provider *D*. We can now see that provider *B* is slightly better than provider *A*, and much better than the other two, given the information available. Furthermore, the result is insensitive to changes in input values, rendering it stable. The advantages of solving problems in this way become even clearer when dealing with large problems, but this example demonstrates the principles at work.

Sensitivity analyses

Uncertainty is inherent in virtually all information in real decision situations. It is ensured that the requirements concerning precision in the input data of the method above are as minimal as possible, while still enabling a decision outcome. This is achieved by employing cardinal ranking instead of numerical input, and forming uncertainty intervals around the weights and values. One should therefore investigate how changes in different components affect the final result. We can now investigate the stability of the choice of a strategy (alternative) when the input data change. Here, we primarily investigate the limits within which the weights and values must remain for the decision not to change. This is achieved by allowing the input values to vary between possible realistic values and to investigate how these fluctuations affect the outcome. Thus, the values are systematically varied up and down.

We can analyze this in several ways. For instance, we can study the stability to investigate the most important values. Often when we specify an interval for a variable, we probably do not believe in all values of the intervals equally, and rather may believe less in values closer to the boundaries of the intervals. Values near the boundaries are nevertheless added to the intervals to cover everything that we perceive as being possible given the uncertainty of the decision problem, but with an indication of the strengths with which we actually believe in the different values. Figure 4 exemplifies a possible belief in a weight of a criterion, where there emphasis is on the middle values of the interval.

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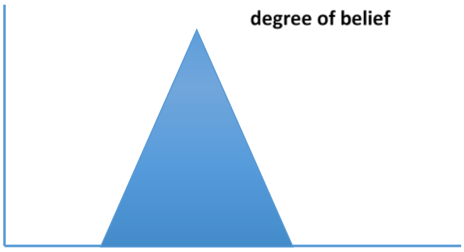

Figure 4. Beliefs in different values

In analysing the decision solution and its stability, we want to know what the situation looks like if we gradually reduce the interval parts in which we have least belief and focus on those that we believe in the most. We call this *contraction*, and it is realised systematically with all of the variables involved.[12] Figure 5 displays the changes in the expected values for a particular alternative during these analyses.

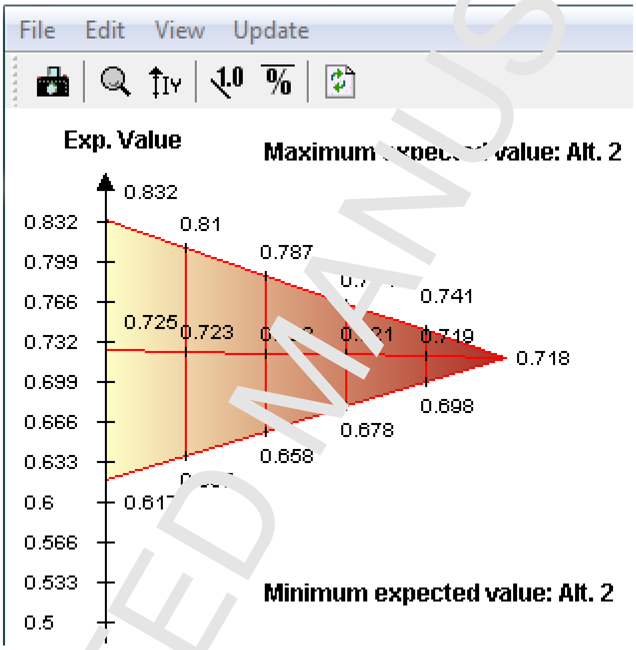

Figure 5. Contraction analysis for provider *B*.

We can see how the initial expected value for provider *B* (at 0 % contraction) ranges approximately between 0.617 and 0.832. At 40 % contraction, it lies approximately between 0.658 and 0.787, and is 0.718 at full contraction (the most likely expected value).

The same analysis can be made for a pair of alternatives. Figure 6 shows how two providers relate to one another. A slightly simplified reading is that the greater the proportion of the triangle found above the x-axis, the better the strategy, and vice versa for the other strategy. Regarding the example, we therefore see that the decision is not totally stable (relatively sensitive to input data), but that provider *B* is better than provider *A* given the current information.

[12] There are also functions to study the sensitivity of each variable separately, so-called tornado diagrams. In such cases, each variable's contribution can be studied, but a treatment of such functions is out of scope of this article.

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When the triangle area is fairly centred with respect to the x-axis, it can still be difficult to determine a recommended strategy due to similarities or significant overlaps, and so we may seek to collect more information. We can then (for instance) use tornado diagrams to investigate which information is most important to the decision and to establish how best to allocate resources for further investigation. In short, an overview of the effectiveness of the respective strategies can be gained by examining how much of the area is located above and below the x-axis. As can be seen from Figure 6, provider *B* is slightly better than provider *A* in this respect. However, there is more to the picture. Further calculating the more detailed distribution of the belief mass (Ekenberg et al., 2005) yields a percentage of the mass above or below the x-axis, i.e. the percentage of belief supporting either one alternative or the other. In Figure 6, even though the triangle is fairly centred, most of the belief mass resides with alternative 2, which is provider *B*.

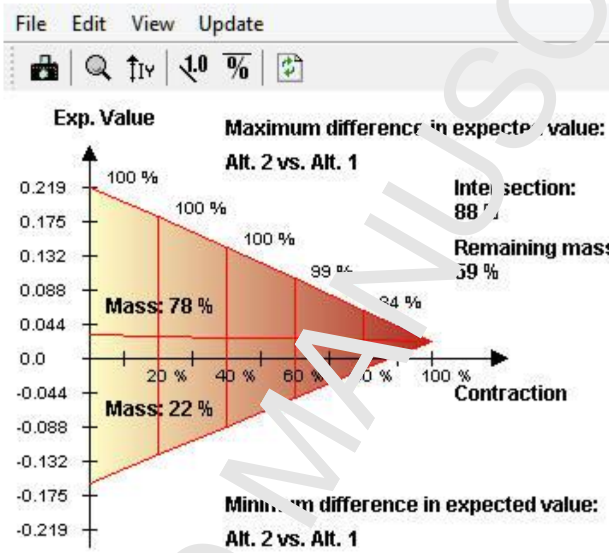

Figure 6. Comparing the two strategies.

In summary, a holistic perspective of the entire decision situation is displayed in Figure 7. The respective bars show the extent to which the various criteria contribute to the final values of the strategies (alternatives). For instance, the criterion *Functionality* contributes significantly to the value of provider *A*, but not as much to providers *C* or *D*.

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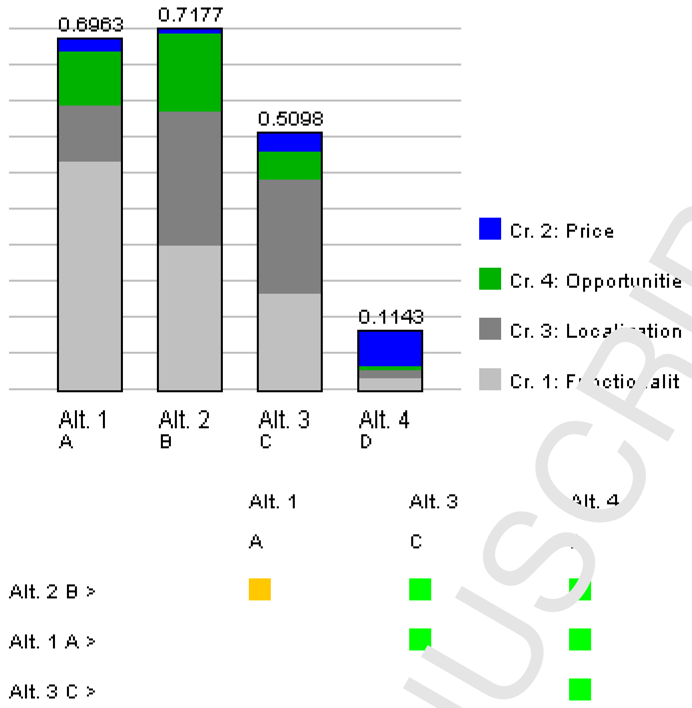

Figure 7. Comparing all strategies at the same time.

The figure also shows the confidence levels of the results, based on the distribution of the belief mass. It is clear that the differences between the providers are significant and that provider *B* is the best with mild confidence[13] that provider *A* comes second, followed by provider *C* with high confidence, and finally provider *D* with a very low value and also with high confidence. Provider *B* should therefore be selected if we have no more information. However, given that the confidence in the separation between provider *B* and *A* is lower (we saw in Figure 6 that it is 78 %), it might be worth investigating if more information exists. It is clear that neither *C* nor *D* is a candidate to consider. In the calculations, we have used the software DecideIT (version 3.0), which can be freely used as long as it is for non-commercial purposes (Preference, 2018). A simplified software for similar purposes is Policy Analysis Tool (POLA) (Larsson et al., 2018), which is used for example by Swedish municipalities for infrastructure investments. The latter is also free to use with the same restrictions applying (POLA, 2018).

6. Comparison

The proposed method can be compared and validated in two steps. In the first step, the proposed P-SWING method is compared to the same decision analytical method without P-SWING, and in the second step, the latter is compared with other well-known methods such as SMART and AHP. The P-SWING method was conceptually validated in focus group discussions, where the inadequacy of standard SWING and non-SWING methods were discussed. Both the conceptual functionality and the actual process implied by the method were endorsed by the vast majority of focus group participants and favoured over both the SWING and non-SWING methods of eliciting and validating criteria weights. This ensured that the method was implemented in software and run on a number of test cases, one being the example highlighted in the previous section. The example in section 5 is built on a real-life

[13] Confidence here is based on the concept of support level, stating the amount of values where one alternative is better than another. For example, if alternative P is better than Q for 22 % of the assigned values and Q is better than P for 78 % of the values, then we should choose Q over P. Simply stated, it is much more likely that alternative Q is best if we do not have more information than already provided.

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case where a 120 MEUR building was to be acquired in a real-estate procurement process with options for acquiring existing buildings as well as constructing from scratch.

The most important difference is the quality assurance enabled by P-SWING. The input rankings of the criteria are much more reliably validated through the cross-validation performed by the partial swinging. The decision maker is given the opportunity to perform an extra quality assurance and enhancement step. As a result, the decision process outcome is further verified. In the example (see Figure 8), it can be seen that the two highest ranked alternatives, A and B, move closer together as a result of improved input quality.

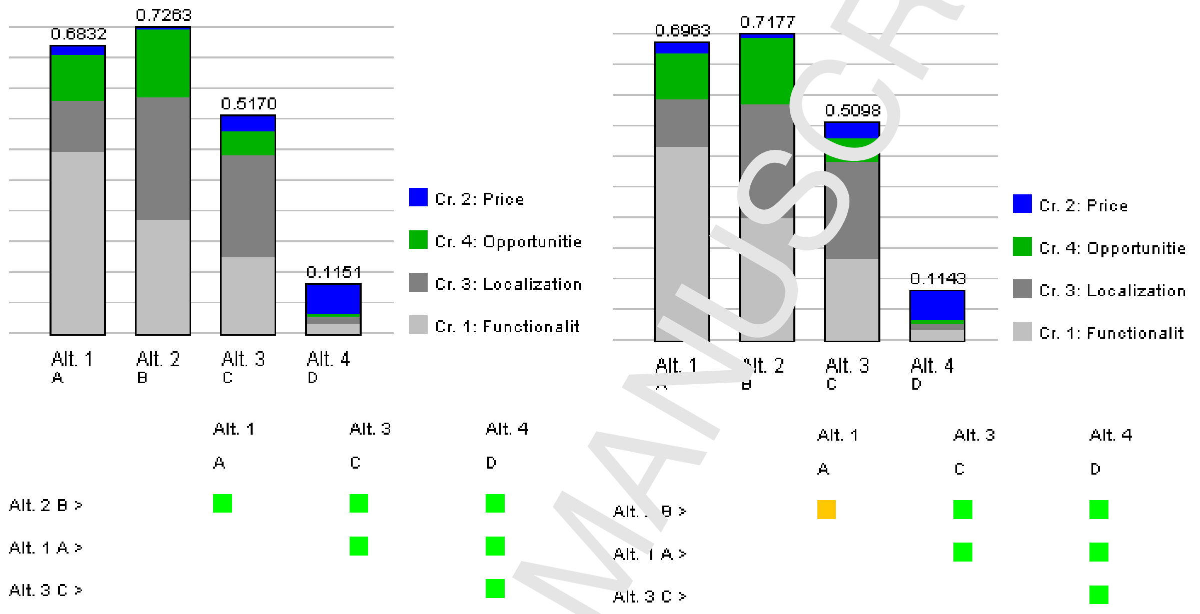

Figure 8. Standard SWING (left) and P-SWING (right).

Having established P-SWING as an additional quality measure for ranking MCDM methods such as CAR, the next step is to place it among other types of methods. In Danielson and Ekenberg (2016) is presented a thorough investigation of three dominating classes of MCDN methods: scoring methods, ranking methods and pairwise comparative methods. The paper establishes ranking methods as one of the major classes of methods, being preferable in a large real-life investigation to the other two classes both on the grounds of performance and user experiences and satisfaction. The addition of a quality assurance step in ranking methods could serve as a quality enhancer, as proposed in the focus groups that led to the design of P-SWING.

7. Conclusions

The elicitation methods that are today available in MCDM are often too cognitively demanding for normal real-life decision makers, and there is a clear need for weighting methods that do not require formal decision analysis knowledge. The SMART method and SWING weighting (in their varieties) are highly beneficial for actual decision-making, in spite of the fact that they are occasionally difficult to understand. Following experiments with 139 participants, we advise against the use of pure swing-style elicitation techniques on the grounds of misunderstanding and misinterpreting the relative nature of swing weights, unless they are amended with additional procedural components to aid understanding. The main contribution of this article is the modification of the SWING family of elicitation techniques and the suggestion of a refined method – the P-SWING method – that allows for intermediate comparisons as well as avoiding synthetic constructs in order to facilitate understanding. In this way, the quality of the weight elicitation can be improved, i.e. it is first and foremost a quality assurance method, an issue of considerable importance according to the focus group discussions. We have also demonstrated how this can be combined with an extension of an existing method and the enhanced DecideIT tool as part of an integrated decision process.

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